Duality and field redefinition in three dimensions

Duality and field redefinition in three dimensions

Márcio A M Gomes and R R Landim

Departamento de Física, Universidade Federal do Ceará, CP 6030, 60470-455 Fortaleza-Ce, Brazil

E-mail: gomes@fisica.ufc.br and renan@fisica.ufc.br

Abstract
We analyse local field redefinition and duality for gauge field theories in three dimensions. We find that both Maxwell–Chern–Simons and the self-dual models admit the same field redefinition. Maxwell–Proca action and its dual also share this property. We show explicitly that a gauge-fixing term has no influence on duality and field redefinition.

PACS numbers: 11.15.−q, 11.10.Kk

1. Introduction

In this paper, we investigate the relationship between duality and field redefinition, in a sense introduced in [1] and [2], for Maxwell–Chern–Simons and Maxwell–Proca models in three spacetime dimensions.

It is widely known that a sufficient condition for duality is the existence of a global symmetry in the original action and one finds the dual model 'gauging' this symmetry [3]. The original model is now seen as a gauge-fixed version of its dual model. One secure prescription for achieving that is 'gauge embedding' procedure [4].

This formalism can be used to derive the well-known duality between the self-dual model (SD) and the Maxwell–Chern–Simons action (MCS) [5]. Some questions are natural consequences of this fact. The first is whether this duality implies that both models will have the same field redefinition. The answer is yes, and we prove this. A second question is if this property holds for some other dual gauge models. To provide an example, we deal with the Maxwell–Proca action (MP) and its dual to show that they indeed have the same field redefinition. A more rigorous treatment about this apparent connection between duality and field redefinition is given in [6].

This paper is organized as follows. In section 2, we prove that the SD and MCS models have the same field redefinition. Section 3 is devoted to establishing the same fact for MP and its dual. In the appendix, we detail the rules for functional calculus with differential forms.

We use the following conventions: the metric adopted is $(- + +)$. We omit the wedge product symbol $\wedge$ and use the inner product of two p-forms $(\omega_p, \eta_p) = \int \omega_p \wedge^* \eta_p$. $*$ denotes the usual duality Hodge operator.

2. Field redefinitions on Maxwell–Chern–Simons and self-dual models

Twenty years ago, Deser and Jackiw [5] established duality between (SD) action

$$S_{SD} = \int \left(\frac{m^2}{2} A^* A + \frac{1}{2} m A \, dA \right) = \frac{m^2}{2}(A, A) - \frac{m}{2}(A, {}^* dA), \tag{2.1}$$

and topologically massive MCS action

$$S_{MCS} = \int \frac{1}{2} \, dA^* \, dA - \frac{m}{2} A \, dA = \frac{1}{2}(dA, dA) + \frac{m}{2}(A, {}^* dA), \tag{2.2}$$

following the clue that both models involve a massive vectorial field in three dimensions and violate parity. They derive the duality from a master action. Each model is obtained combining equations of motion and this master action. This can be also achieved using gauge embedding formalism [4].

In [1] and [2], Lemes *et al* showed that the Chern–Simons term can be seen as a generator for MCS model, through a field redefinition. We set the redefinition for

$$\frac{1}{2}(dA, dA) + \frac{m}{2}(A, *dA) = \frac{m}{2}(\hat{A}, *d\hat{A}), \tag{2.3}$$

where

$$\hat{A} = A + \sum_{i=1}^{\infty} \frac{A_i}{m^i}. \tag{2.4}$$

It was shown in [7] that

$$\delta A_i = 0, \tag{2.5a}$$

$$d^\dagger A_i = 0, \tag{2.5b}$$

and that A_i depends linearly on A in the following way:

$$A_i = \alpha_i (*d)^i A. \tag{2.6}$$

We can also look at field redefinition (2.4) as having an operatorial nature[1],

$$\hat{A} = \mathcal{O} A, \tag{2.7}$$

where $\mathcal{O}$ is an operator defined by

$$\mathcal{O} = \left(1 + \sum_{j=1}^{\infty} \frac{\alpha_j (*d)^j}{m^j} \right). \tag{2.8}$$

The operator $\mathcal{O}$ has the following properties:

$$(\mathcal{O} A, B) = (A, \mathcal{O} B), \tag{2.9a}$$

$$[\mathcal{O}, *d] = 0, \tag{2.9b}$$

---

[1] Let us comment about a functional formulation with this field redefinition. Since this redefinition is linear, the functional measure gains a factor which does not depend on the field and hence is factorable from the functional.

for any 1-forms A and B. So, we can write the field redefinition for the MCS model in the form

$$\frac{1}{2}(dA, dA) + \frac{m}{2}(A, *dA) = \frac{m}{2}(\mathcal{O}A, *d\mathcal{O}A). \tag{2.10}$$

Taking the functional derivative with respect to A on both sides of this equation we obtain

$$\left(\frac{*d}{m} - 1 + \mathcal{O}^2\right) *dA = 0. \tag{2.11}$$

Since the operator $\mathcal{O}$ depends only on $*d$, we must have

$$\mathcal{O}^2 = 1 - \frac{*d}{m}, \tag{2.12}$$

and consequently

$$\mathcal{O}^2 A = A - \frac{*d}{m}A. \tag{2.13}$$

Integrating with respect to A we get

$$\tfrac{1}{2}m^2(A, A) - \tfrac{1}{2}m(A, *dA) = \tfrac{1}{2}m^2(\hat{A}, \hat{A}). \tag{2.14}$$

So the field redefinitions to MCS and its dual SD model are the same and the SD model can be reset in a pure mass-like term. From (2.12) we can obtain the form of the operator $\mathcal{O}$ by expansion in power series in $*d/m$

$$\mathcal{O} = 1 - \sum_{j=1}^{\infty} \frac{(2j)!}{2^{2j}(2j-1)(j!)^2} \left(\frac{*d}{m}\right)^j. \tag{2.15}$$

Comparing (2.7) and (2.15), we find out a formula for the coefficients α_j

$$\alpha_j = -\frac{(2j)!}{2^{2j}(2j-1)(j!)^2}, \tag{2.16}$$

that furnishes the same coefficients found in [8].

We can directly invert the field redefinition

$$A = \mathcal{O}^{-1}\hat{A}, \tag{2.17}$$

with

$$\mathcal{O}^{-1} = \left(1 - \frac{*d}{m}\right)^{-1/2} = \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}(k!)^2} \left(\frac{*d}{m}\right)^k, \tag{2.18}$$

that are also in complete agreement with those in [8].

The series expansion

$$(1+x)^{1/2} = \sum_{j=1}^{\infty} (-1)^j \alpha_j x^j, \tag{2.19}$$

converges only for $|x| < 1$ and $x = 1$, but this is not a problem since one can easily extend the convergence for other intervals by analytic continuation.

We can change our way of thinking and consider (2.11) had been obtained from (2.13) after application of $*d$. Thus redefined MCS is generated by the redefined SD model, both presenting the same field redefinition. One can reasonably argue what would one get applying $*d$ twice on (2.13) and integrating on field A. We obtain a new gauge invariant model

$$-\frac{1}{2}(dA, d * dA) + \frac{m}{2}(dA, dA) = \frac{m}{2}(d\hat{A}, d\hat{A}). \tag{2.20}$$

By construction the field redefinition of (2.20) is exactly that for the SD and MCS models. In fact, these models belong to a same equivalence class [6].

An interesting point is that adding a gauge-fixing term to the SD model has no influence on field redefinition since it does not spoil duality. Consider the self-dual model plus a Landau gauge, fixing term

$$S^{(0)} = \int \left(\frac{m^2}{2} A_\mu A^\mu + \frac{\alpha}{2} (\partial_\mu A^\mu)^2 - \frac{m}{2} \epsilon_{\mu\nu\rho} A^\rho \partial^\mu A^\nu \right) d^3x. \tag{2.21}$$

Following the gauge embedding prescription [4], the first iterative action is

$$S^{(1)} = S^{(0)} - \int (m^2 A_\mu - \alpha \partial_\mu (\partial A) - m \epsilon_{\mu\nu\rho} \partial^\nu A^\rho) B^\mu \, d^3x, \tag{2.22}$$

and it follows that

$$\delta S^{(1)} = -\delta \int \frac{m^2}{2} B_\mu B^\mu \, d^3x - \delta \int \frac{\alpha}{2} (\partial_\mu B^\mu)^2 \, d^3x. \tag{2.23}$$

Then, the invariant action is

$$S^{(2)} = S^{(1)} + \int \left(\frac{m^2}{2} B_\mu B^\mu + \frac{\alpha}{2} (\partial_\mu B^\mu)^2 \right) d^3x. \tag{2.24}$$

The equation of motion for B_μ shows that $\partial_\mu B^\mu = \partial_\mu A^\mu$ and $m^2 B_\mu = m^2 A_\mu - m \epsilon_{\mu\nu\rho} \partial^\nu A^\rho$. This points to the Maxwell–Chern–Simons as being the dual model no matter what the gauge-fixing term is.

3. Field redefinitions on Maxwell–Proca and its dual

Another model which shares the same field redefinition with its dual is MP,

$$S_{\text{MP}} = \frac{m^2}{2}(A, A) - \frac{1}{2}(*dA, *dA) = \frac{m^2}{2}(A, (1 - (*d/m)^2)A). \tag{3.1}$$

We can find its dual following the gauge embedding procedure:

$$S_{\text{dual–MP}} = \frac{1}{2}(*dA, *dA) - \frac{1}{2m^2}(*d*dA, *d*dA). \tag{3.2}$$

Just like the SD model, MP can be redefined to a pure mass-like term, namely

$$S_{\text{MP}} = \frac{m^2}{2}(\hat{A}, \hat{A}) = \frac{m^2}{2}(OA, OA), \tag{3.3}$$

where O also satisfies properties (2.9). From (3.1) and (3.3), we see that O is nothing but

$$O = (1 - (*d/m)^2)^{1/2}. \tag{3.4}$$

Repeating the steps in section 2, we are left with

$$\left[O^2 - 1 + \left(\frac{*d}{m} \right)^2 \right] A = 0. \tag{3.5}$$

Applying $(*d)^2$ and integrating in A:

$$S_{\text{dual–MP}} = \frac{1}{2}(*dA, *dA) - \frac{1}{2m^2}(*d*dA, *d*dA) = \frac{1}{2}(*d\hat{A}, *d\hat{A}). \tag{3.6}$$

Observe that applying $*d$ in (3.5) and integrating, one gets another theory with same field redefinition as MP that is not its dual. We clear this point in [6].

4. Conclusion

In this work, we analysed the aspects of field redefinition and duality of gauge theories in three dimensions. We showed that the MCS and SD models can be rewritten as a unique term with the same field redefinition. Indeed, both models are members of a equivalence class of gauge theories. Let us remark that such a redefinition is local and has a closed expression in operator $*d$, e.g, see equation (2.12). The expansion in power of $*d$ is formal and only makes sense when operating on a gauge field. Our results clarify some aspects of duality in three dimensions. In this approach, the gauge-fixing term drops out the dual theory.

Acknowledgment

Appendix. Functional calculus with differential forms

We start by defining the functional derivative of a p-form $A(x)$. Let the inner product of two p-forms be given by

$$(A, B) = \int A(x) * B(x) = \int \frac{1}{p!} A(x)_{\mu_1..\mu_p} B(x)^{\mu_1..\mu_p} \, d^D x, \tag{A.1}$$

where we are considering a flat manifold. Taking the functional derivative with respect to $A(y)_{\nu_1..\nu_p}$ on both sides of (A.1), one gets

$$\frac{\delta}{\delta A(y)_{\nu_1..\nu_p}}(A, B) = B(y)^{\nu_1..\nu_p}. \tag{A.2}$$

Then we define the functional derivative of a p-form $A(x)$ as

$$\frac{\delta}{\delta A(y)}(A, B) = B(y). \tag{A.3}$$

We also define the Dirac delta p-form

$$\left(\delta_p^D(x - y), B(x)\right) = B(y), \tag{A.4}$$

then

$$\frac{\delta A(x)}{\delta A(y)} = \delta_p^D(x - y). \tag{A.5}$$

From definition (A.4), we can get the explicit form of $\delta_p^D(x - y)$ in terms of the usual Dirac delta function:

$$\delta_p^D(x - y) = \frac{1}{p!}\delta^D(x - y) g_{\mu_1\nu_1}..g_{\mu_p\nu_p} \, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \otimes dy^{\nu_1} \wedge \cdots \wedge dy^{\nu_p}. \tag{A.6}$$

Clearly, $\delta_0^D(x - y) = \delta^D(x - y)$. Note that a p-form is defined in a point of the co-tangent space, therefore two p-forms in different points always commute since they are defined in different co-tangent spaces, i.e., $B(x)B(y) = B(y)B(x)$.

References

[1] Lemes V E R, Linhares de Jesus C, Sasaki C A G, Sorella S P, Vilar L C Q and Ventura O S 1998 *Phys. Lett*. B **418** 324
[2] Lemes V E R, Linhares de Jesus C, Sorella S P, Vilar L C Q and Ventura O S 1998 *Phys. Rev.* D **58** 045010
[3] Quevedo F 1998 *Nucl. Phys*. B **61A** 23–41
[4] Ilha A and Wotzasek C 2001 *Nucl. Phys*. B **604** 426
    Ilha A and Wotzasek C 2001 *Phys. Lett*. B **519** 169
[5] Deser S and Jackiw R 1984 *Phys. Lett*. B **139** 2366
[6] Marcio A M Gomes and Landim R R 2004 *Preprint* hep-th/0405266
[7] Márcio A M Gomes and Landim R R 2004 *Preprint* hep-th/0403184
[8] Lemes V E R, Linhares C A, Sorella S P and Vilar L C Q 1999 *J. Phys. A: Math. Gen.* **32** 2469